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BRIEF COMMUNICATION

THE EFFECT OF SEPARATION ON DRAG AND TORQUE IN STOKES FLOW

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1. INTRODUCTION

This paper assesses the effect which Stokes flow separation has on the drag and torque on a fixed body. In Dorrepaal *et al.* (1976) the axisymmetric streaming Stokes flow past a spherical cap was found to produce separation on the cap for all cap angles α, $0 < \alpha < \pi$. It is shown here that if the surface area is kept constant the maximum drag occurs in the limiting case of the disk where there is no separation and decreases monotonically with increasing α until the minimum is attained in the case of a sphere. Clearly the effect of separation in this situation is to decrease the drag on the cap.

Dorrepaal (1976) first discussed the separation on a spherical cap in asymmetric streaming Stokes flow and gave general expressions for the drag and torque on the cap. If the area of the cap is kept constant the drag increases with α until $\alpha = 61.40°$, then decreases with increasing α. Separation commences at 60.90° and clearly the effect of separation is drag reduction. The torque on the cap is also considered together with the flow produced by a two dimensional eccentric bearing. In both cases a similar conclusion is borne out.

2. DRAG ON A SPHERICAL CAP

Consider the set of all spherical caps having constant surface area A. If a cap of radius c and angle α belongs to this set then

$$A = 2\pi c^2(1 - \cos\alpha). \qquad [2.1]$$

Let D be the Stokes drag experienced by the cap as it moves parallel to its axis through a quiescent fluid. From Dorrepaal *et al.* (1976) we have

$$D = Uc\rho\nu(6\alpha + 8\sin\alpha + \sin 2\alpha), \qquad [2.2]$$

where U is the cap's velocity and ρ and ν are fluid density and viscosity respectively. We can eliminate c from (2.2) using (2.1) and then define $D_s(\alpha)$ to be the normalization of D such that $D_s(0) = 1$. Thus we have

$$D_s(\alpha) = \frac{6\alpha + 8\sin\alpha + \sin 2\alpha}{32\sin\frac{1}{2}\alpha}, \quad 0 < \alpha < \pi. \qquad [2.3]$$

When $\alpha = \pi$, the cap is a sphere and in the limiting case $\alpha \to 0$, $c \to +\infty$ with $\alpha c = b$, the cap

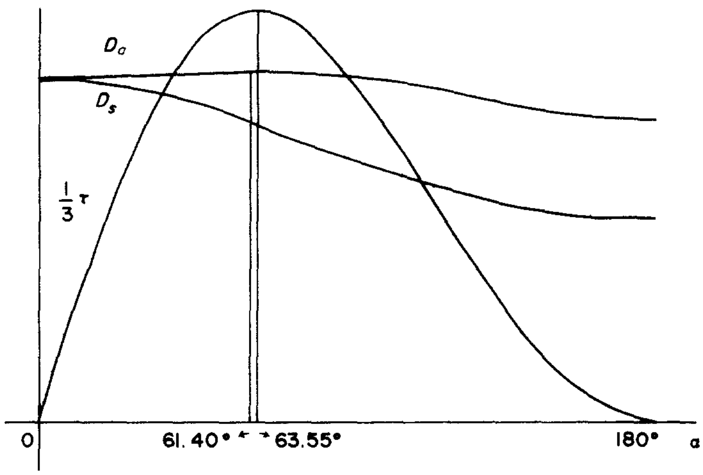

Figure 1. Drag and torque vs cap angle α.

degenerates to a disk of radius b. Figure 1 shows that D_s attains its maximum value in the limiting case $\alpha = 0$.

Dorrepaal *et al.* (1976) have shown that the axisymmetric Stokes flow past a spherical cap separates for all angles $\alpha > 0$. In the case of the disk however, no wake is present. D_s is a maximum therefore when the axisymmetric flow is about to separate. As α increases and the wake develops, D_s decreases monotonically.

In a similar way let $D_a(\alpha)$ be the normalized Stokes drag experienced by a cap which moves perpendicular to its axis. Once again we consider only caps with surface area A. From Dorrepaal (1976) we have

$$D_a(\alpha) = \frac{3}{64 \sin\frac{1}{2}\alpha}\left\{6(\alpha + \sin\alpha) - \frac{8}{3}\frac{\sin^2\alpha \cos^4\frac{1}{2}\alpha}{\alpha + \sin\alpha}\right\}. \quad [2.4]$$

Figure 1 shows that D_a has a maximum when $\alpha = 61.40°$. Dorrepaal (1976) has shown that the asymmetric Stokes flow first separates when $\alpha = 60.90°$. Just as in the axisymmetric case the drag attains a maximum when the flow begins to separate. As the wake grows larger with increasing α, D_a decreases.

3. TORQUE ON A SPHERICAL CAP

When a cap moves perpendicular to its axis in a quiescent fluid, it experiences a torque about its centre of mass. In the limiting cases of a disk and sphere the torque is zero. A cap having area A experiences a torque which from Dorrepaal (1978) is proportional to

$$\tau(\alpha) = \frac{1}{\sin^2\frac{1}{2}\alpha}\left\{(3\alpha - \sin\alpha)\cos^2\frac{1}{2}\alpha - \frac{2}{3}\frac{(5 + \cos\alpha)\sin^2\alpha \cos^4\frac{1}{2}\alpha}{\alpha + \sin\alpha}\right\}. \quad [3.1]$$

The curve of τ vs α has a maximum at $\alpha = 63.55°$. As with the asymmetric drag the torque is greatest when the flow about the cap has just begun to separate.

4. ECCENTRIC BEARING

Consider now a bounded Stokes flow which exhibits separation. One example of such a flow is the eccentric bearing investigated by Jeffery (1922) and Wannier (1950). A small cylinder rotates inside a larger stationary cylinder causing the fluid between the cylinders to rotate. The cylindrical radii are a_1 and a_2 ($a_1 < a_2$) and the distance e between the axes of the cylinders is a parameter of the problem. The case $e = 0$ gives a concentric configuration and when $e = a_2 - a_1$, the two cylinders are tangent.

The case $a_1 = 1$, $a_2 = 2$ is typical. The torque on the outer cylinder is proportional to

$$T = \frac{\sinh^2 \alpha_1[(\alpha_1 - \alpha_2)\coth(\alpha_1 - \alpha_2) - 1]}{(\alpha_1 - \alpha_2)(\sinh^2 \alpha_1 + \sinh^2 \alpha_2) - 2\sinh \alpha_1 \sinh \alpha_2 \sinh(\alpha_1 - \alpha_2)}, \quad [4.1]$$

where

$$\cosh \alpha_1 = \frac{3 - e^2}{2e} \text{ and } \sinh \alpha_2 = \frac{1}{2}\sinh \alpha_1 .$$

From figure 2 it is seen that T is a monotonically decreasing function of e which vanishes in the limit $e \to 1$. It can be shown from Wannier (1950) that a region of reverse flow first appears when $e = 0.32$ and grows as e increases.

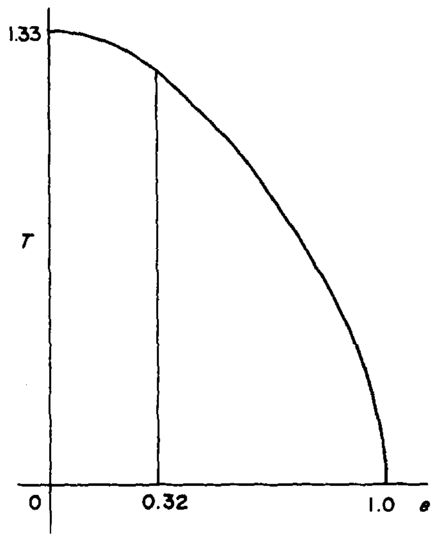

Figure 2. Torque vs distance between cylindrical axes.

5. CONCLUSIONS

Both the axisymmetric and asymmetric Stokes flows past a spherical cap exhibit separation along the concave surface of the cap. If one considers the set of all caps having constant surface area, then the member of this set experiencing the maximum drag in either case is the cap which is just concave enough to induce separation. As α increases beyond this value the wake becomes larger and the drag actually drops. In the asymmetric flow a similar result holds for the torque about the cap's centre of mass. These observations suggest that as a body deforms keeping its surface area constant, the formation and development of Stokes wakes have a streamlining effect dynamically thus reducing drag and torque.

The case of the eccentric bearing is different and yet a similar principle holds—namely, separation of the flow serves to reduce the torque on the outer cylinder. When there is no reverse flow the shear stress is in the same direction over the entire circumference of the outer cylinder. But when the flow separates the shear stress along the boundary in the separated region is opposed to that over the rest of the circumference. The result is a reduction in the torque. In the limiting case when the two cylinders are tangent, the shear stress is singular at the point of contact and the contribution to the torque from this singularity exactly cancels the contribution from the region of reverse flow. Thus the torque vanishes when the cylinders are in contact.

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APPENDIX

Since the shear stress on the boundary of the cap changes sign when separation or reverse flow occurs and stress is force per unit area it is necessary to maintain the total area of the cap constant when examining the variation of the force on the cap. If the cross-section of the cap is kept constant towards the mainstream flow then it is found that the drag on the cap increases with the angle of the cap, reaching a maximum in the case of a sphere.